Math 53, Discussion 105

Wednesday, April 15, 2020 9:31 AM

16.3 (Fundamental theorem for line integrals) and 16.4 (Green's theorem)

Recall some formulae:

$\sqrt{(x')^2+(y')^2}$

(1) $$\int_C f(x,y)\,ds = \int_a^b f(r(t))\|r'(t)\|\,dt$$

(2) $$\int_C f(x,y)\,dx = \int_a^b f(r(t))x'(t)\,dt$$

(3) $$\int_C f(x,y)\,dy = \int_a^b f(r(t))y'(t)\,dt$$

(4) $$\int_C \langle P,Q,R\rangle \cdot dr = \boxed{\int_C Pdx + Qdy + Rdz}$$

$\vec{F}$

Example: compute $\int_C xyzds$ with $r(t) = \langle 2\sin t, t, -2\cos t\rangle$, $0 \le t \le \pi$

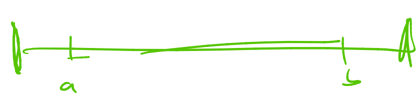
$$\|r'(t)\| = \sqrt{4\cos^2(t) + t^2 + 4\sin^2(t)} = \sqrt{4+t^2}$$

$$\int_0^\pi (2\sin(t))(t)(-2\cos(t))\sqrt{4+t^2}\,dt$$

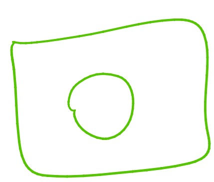
Theorem: (Multivar. fundamental theorem of calc)

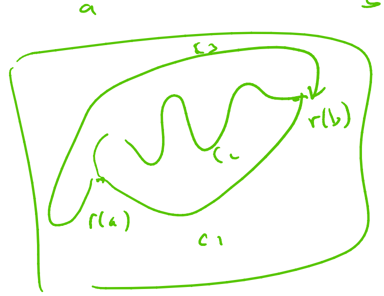
$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a)) \longleftrightarrow \int_a^b f'(x)\,dx = f(b) - f(a)$$

Not all vector fields are conservative



curve going from $r(a) \to r(b)$





Theorem: if $F = \langle P, Q\rangle$ is conservative, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ (coverse true if in a simply connected region)

$F = \langle \cos \ldots, 2 \ldots \rangle$ is this ... line?

simply connected region)

$F = \langle 6y, 7x \rangle$ is this conservative?

$6 \neq 7$

if F is cons. $\boxed{F = \langle f_x, f_y \rangle}$

$f_{xy} = f_{yx}$

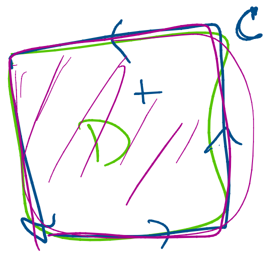
$6y \neq f_x$ for any f.

$7x \neq f_y$

Theorem: $\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

(Green's Theorem)

C is boundary of D, counter clockwise



16.3

Exercises:

1. Compute $\int_C F \cdot dr$ for $F = \langle y^2 cosz, 2xy\cos z, -xy^2 \sin z \rangle$ for $C = \langle t^2, t+1, 2t-1 \rangle$ for $0 \leq t \leq 1$ (hint: this is conservative)
2. Determine if the following vector fields are conservative (if so, find f such that $\nabla f = F$

$\langle 2x - 3y, -3x + 4y - 8 \rangle$

$\langle e^x \cos y, e^x \sin y \rangle$

16.4

3. Evaluate the integral directly, then with Green's theorem $\oint_C (x - y)dx + (x+y)dy$ with C the circle at the origin and radius 2
4. Evaluate the integral using Green's theorem: $\int_C \cos y\, dx + x^2 \sin y dy$ with C the rectangle with vertices (0,0), (5,0), (5,2), (0,2)

3) $\underbrace{\oint P dx + Q dy}_{F} = \iint_{\text{circle}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$= \iint 1 - (-1)\, dA = 2(4\pi) = 8\pi$

$\int_0^2 \int_0^5 \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

2) a) $\frac{1}{\partial y} \stackrel{?}{=} \frac{\quad}{\partial x}$

$-3 \stackrel{?}{=} -3$ ✓

$$\boxed{x^2 + 2y^2 - 3xy - 8y}$$

b) $\frac{\partial P}{\partial y} = -e^x \sin(y)$

$\frac{\partial Q}{\partial x} = e^x \sin(y)$ $\Rightarrow$ not conservative.

1) $f = x y^2 \cos(z)$

$f_x = y^2 \cos(z)$

$y^2 \cos(z) = f_x$

$f = x y^2 \cos(z) + g(y,z)$

$r(0) = (0,1,-1)$

$(1,2,1) \leftarrow r(1)$

$\int_C F \cdot dr = f(1,2,1) - f(0,1,-1)$

$= 4\cos(1)$